Najtoplije zahvaljujem **prof. Luki Čelikoviću** na dozvoli da skeniram sažetak predavanja
"Primjena kompleksnih brojeva u trigonometriji"
i objavim na svojim web stranicama.

Isprintajte obostrano i presavinite na pola.

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Iz (***) slijedi:

$$\cos x + i \sin x = (1/2) \pm (i\sqrt{3}/2) \Rightarrow (\cos x = 1/2 \wedge \sin x = \pm i\sqrt{3}/2)) \Rightarrow$$
$$\Rightarrow x_{2,3} = \pm \pi/3 + 2k\pi,\ k\in\mathbb{Z}.$$

Primjer 5. Transformirati umnožak potencija trigonometrijskih funkcija

$$\sin^4\alpha \cdot \cos^3\alpha$$

u trigonometrijski polinom.

Rješenje:

Primjenom (1), (2) i (5) izlazi:

$$\sin^4\alpha\cdot\cos^3\alpha = ((z^2 - 1) / 2iz)^4 \cdot ((z^2 + 1) / 2z)^3$$
$$= (z^{14} - z^{12} - 3 z^{10} + 3 z^8 + 3 z^6 - 3 z^4 - z^2 + 1) / 128z^7$$
$$= (1/64)\cdot((1/2)(z^7 + 1/z^7) - (1/2)(z^5 + 1/z^5) - 3\cdot(1/2)(z^3 + 1/z^3) + 3\cdot(1/2)(z+1/z))$$
$$\overset{(5),(1)}{=} (1/64)\cdot(\cos 7\alpha - \cos 5\alpha - 3 \cos 3\alpha + 3 \cos \alpha)$$
$$= (1/64)\cdot\cos 7\alpha - (1/64)\cdot\cos 5\alpha - (3/64)\cdot\cos 3\alpha + (3/64)\cdot\cos \alpha$$

Zadaci za vježbu:

1. Izračunati zbroj $S = \cos\alpha + \cos 3\alpha + \cos 5\alpha + \dots + \cos (2n-1)\alpha,\ n\in N$.
   (Rezultat: $S = \sin 2n\alpha \ / \ 2 \sin \alpha$).

2. Izračunati zbroj $S = \sin\alpha + \sin 2\alpha + \sin 3\alpha + \dots + \sin n\alpha,\ n\in N$.
   (Rezultat: $S = (\sin ((n+1)/2)\alpha \ \sin (n\alpha/2)) \ / \ \sin (\alpha/2)$).

3. Izračunati umnožak $\Pi = \cos (\alpha/2)\cdot\cos (\alpha/2^2)\cdot\cos (\alpha/2^3)\cdot \dots \cdot\cos (\alpha/2^n),\ n\in N$.
   (Rezultat: $\Pi = \sin\alpha \ / \ 2^n \sin (\alpha/2^n)$).

4. Riješiti jednadžbu $\cos 4x + \sin^2 3x = 1$.
   (Rezultat: $x_{1,2} = \pm (\pi/12) + (k\pi/2),\ x_3 = k\pi,\ k\in\mathbb{Z}$).

5. Dokazati trigonometrijski identitet $\sin (\alpha+\beta) \ \sin (\alpha-\beta) = \sin^2\alpha - \sin^2\beta$.

6. Transformirati izraz $\sin^4\alpha$ u trigonometrijski polinom.
   (Rezultat: $\sin^4\alpha = (1/8)\cdot\cos 4\alpha - (1/2)\cdot\cos 2\alpha + 3/8$).

Luka Čeliković

PRIMJENA KOMPLEKSNIH BROJEVA U TRIGONOMETRIJI

U ovom izlaganju bit će govora o primjeni kompleksnih brojeva pri određivanju zbroja i umnoška trigonometrijskih funkcija, dokazivanju trigonometrijskih identiteta, rješavanju trigonometrijskih jednadžbi i pri transformaciji umnoška potencija trigonometrijskih funkcija u trigonometrijski polinom

Neka je $z=a+bi$ $(a,b\in R)$ kompleksan broj čiji je modul $r = |z| = \sqrt{a^2+b^2} = 1$. Tada njegov trigonometrijski oblik glasi

$$z = \cos\alpha + i \sin\alpha \qquad \text{(a)}$$

Za konjugirano kompleksan broj $\bar{z}$ broja z tada izlazi

$$\bar{z} = \cos\alpha - i \sin\alpha, \qquad \text{(b)}$$

pa je

$$z + \bar{z} = 2 \cos\alpha, \qquad \text{(c)}$$
$$z - \bar{z} = 2i\cdot\sin\alpha, \qquad \text{(d)}$$
$$z\cdot\bar{z} = 1, \text{ tj. } \bar{z} = 1/z. \qquad \text{(e)}$$

Iz (c), (d) i (e) izlazi

$$\cos\alpha = (z^2+1)/2z = (1/2)(z+1/z), \qquad (1)$$
$$\sin\alpha = (z^2-1)/2iz = (1/2i)(z-1/z), \qquad (2)$$
$$\mathrm{tg}\,\alpha = (z^2-1)/i(z^2+1), \qquad (3)$$
$$\mathrm{ctg}\,\alpha = i(z^2+1)/(z^2-1). \qquad (4)$$

Primjenom Moivreovih formula

$$z^n = (\cos\alpha + i \sin\alpha)^n = \cos n\alpha + i \sin n\alpha \qquad \text{(f)}$$
$$\bar{z}^n = (\cos\alpha - i \sin\alpha)^n = \cos n\alpha - i \sin n\alpha \qquad \text{(g)}$$

slijedi:

$$\cos n\alpha = (z^{2n} + 1) / 2z^n = (1/2) (z^n + 1/z^n), \qquad (5)$$
$$\sin n\alpha = (z^{2n} - 1) / 2iz^n = (1/2i) (z^n - 1/z^n), \qquad (6)$$
$$\mathrm{tg}\, n\alpha = (z^{2n} - 1) / i(z^{2n}+1), \qquad (7)$$
$$\mathrm{ctg}\, n\alpha = i(z^{2n} + 1) / (z^{2n} - 1). \qquad (8)$$

Za

$$z_1 = \cos\alpha + i \sin\alpha,\ z_2 = \cos\beta + i \sin\beta \qquad \text{(h)}$$

je

$$z_1\cdot z_2 = \cos (\alpha + \beta) + i \sin (\alpha + \beta), \qquad \text{(i)}$$
$$\overline{z_1}\cdot\overline{z_2} = \cos (\alpha + \beta) - i \sin (\alpha + \beta), \qquad \text{(j)}$$
$$z_1 / z_2 = \cos (\alpha - \beta) + i \sin (\alpha - \beta), \qquad \text{(k)}$$
$$\overline{z_1} / \overline{z_2} = \cos (\alpha - \beta) - i \sin (\alpha - \beta). \qquad \text{(l)}$$

Odatle je:

$$\cos (\alpha + \beta) = (z_1^2 z_2^2 + 1) / 2 z_1 z_2, \qquad (9)$$
$$\sin (\alpha + \beta) = (z_1^2 z_2^2 - 1) / 2i z_1 z_2. \qquad (10)$$
$$\mathrm{tg}(\alpha + \beta) = (z_1^2 z_2^2 - 1) / i (z_1^2 z_2^2 + 1), \qquad (11)$$
$$\mathrm{ctg} (\alpha + \beta) = i (z_1^2 z_2^2 + 1) / (z_1^2 z_2^2 - 1), \qquad (12)$$
$$\cos (\alpha - \beta) = (z_1^2 + z_2^2) / 2 z_1 z_2, \qquad (13)$$
$$\sin (\alpha - \beta) = (z_1^2 - z_2^2) / 2i z_1 z_2. \qquad (14)$$
$$\mathrm{tg}(\alpha - \beta) = (z_1^2 - z_2^2) / i (z_1^2 + z_2^2), \qquad (15)$$
$$\mathrm{ctg} (\alpha - \beta) = i (z_1^2 + z_2^2) / (z_1^2 - z_2^2). \qquad (16)$$

Specijalno za

$$z = \cos \pi/n \; + i \sin \pi/n \qquad \text{(m)}$$

iz (f) slijedi:

$$z^{n/2} = i,\; z^n = -1,\; z^{2n} = 1 . \qquad (17)$$

Pokazat ćemo sada na primjerima primjenu formula (1) - (17).

Primjer 1. Bez uporabe tablica, kalkulatora i sl. izračunati zbroj

$$S = \sin (\pi/10) - \sin (3\pi/10) + \sin (5\pi/10) - \sin (7\pi/10) + \sin (9\pi/10).$$

Rješenje:

$$\begin{aligned} S &= \sin (\pi/10) - \sin (3\pi/10) + \sin (5\pi/10) - \sin (7\pi/10) + \sin (9\pi/10) \\ &= (1/2) + 2\,(\sin (\pi/10) - \sin (3\pi/10)) \\ &= (1/2) + 2\,S_1 , \end{aligned}$$

gdje je $S_1 = \sin (\pi/10) - \sin (3\pi/10)$.

Neka je

$$z = \cos (\pi/10) + i \sin (\pi/10).$$

Tada je

$$\sin (\pi/10) \overset{(2)}{=} (z^2 - 1) / 2iz,\quad \sin (3\pi/10) \overset{(6)}{=} (z^6 - 1) / 2iz^3,\quad z^5 \overset{(17)}{=} i,\quad z^{10} \overset{(17)}{=} -1,\quad z^{20} \overset{(17)}{=} 1,$$

pa imamo

$$\begin{aligned} S_1 &= (z^2 - 1) / 2iz - (z^6 - 1) / 2iz^3 \\ &= ((-z^6 + z^4 - z^2 + 1) / 2iz^3)\cdot(z^2 / z^2) \\ &= (-z^8 + z^6 - z^4 + z^2) / 2iz^5 \\ &= (z^2((-z^2)^4 - 1) / (-z^2 - 1))) / (-2) \\ &= (z^{10} - z^2) / 2(z^2 + 1) \\ &= (-1 - z^2) / 2(z^2 + 1) \\ &= - 1/2. \end{aligned}$$

Stoga je $S = (1/2) + 2\cdot(- 1/2)$, tj.

$$\mathbf{S = - 1/2.}$$

Primjer 2. Bez uporabe tablica, kalkulatora i sl. izračunati umnožak

$$\Pi = \cos (\pi/9)\cdot\cos (2\pi/9)\cdot\cos (3\pi/9)\cdot\cos (4\pi/9).$$

Rješenje:

$$\begin{aligned} \Pi &= \cos (\pi/9)\cdot\cos (2\pi/9)\cdot\cos (3\pi/9)\cdot\cos (4\pi/9). \\ &= (1/2) \cos (\pi/9)\cdot\cos (2\pi/9)\cdot\cos (4\pi/9) \\ &= (1/2)\, \Pi_1 , \end{aligned}$$

gdje je $\Pi_1 = \cos (\pi/9)\cdot\cos (2\pi/9)\cdot\cos (4\pi/9)$.

Neka je

$$z = \cos (\pi/9) + i \sin (\pi/9).$$

Tada je:

$$\cos (\pi/9) \overset{(1)}{=} (z^2 + 1) / 2z,\ \cos (2\pi/9) \overset{(5)}{=} (z^4 + 1) / 2z^2,\ \cos (4\pi/9) \overset{(5)}{=} (z^8 + 1) / 2z^4,\ z^9 \overset{(17)}{=} -1,\ z^{18} \overset{(17)}{=} 1,$$

pa imamo:

$$\begin{aligned} \Pi_1 &= ((z^2 + 1) / 2z)\cdot((z^4 + 1) / 2z^2)\cdot((z^8 + 1) / 2z^4) \\ &= ((z^{14} + z^{12} + \ldots + z^2 + 1) / 8 z^7)\cdot(z^2 / z^2) \\ &= (z^{16} + z^{14} + \ldots + z^4 + z^2) / 8 z^9) \\ &= (z^2((z^2)^8 - 1) / (z^2 - 1))) / (-8) \\ &= (z^{18} - z^2) / (-8(z^2 - 1)) \\ &= (1 - z^2) / (-8(z^2 - 1)) \\ &= 1/8. \end{aligned}$$

Stoga je $\Pi = (1/2)\cdot(1/8)$, tj.

$$\Pi = 1/16.$$

Primjer 3. Dokazati trigonometrijski identitet

$$\sin \alpha + \sin 2\alpha + \sin 3\alpha \;=\; (\sin 2\alpha \sin (3\alpha/2)) / \sin (\alpha/2).$$

Rješenje:

Neka je

$$z = \cos (\alpha/2) + i \sin (\alpha/2).$$

Tada je:

$$\sin (\alpha/2) \overset{(2)}{=} (z^2 - 1) / 2iz ,\quad \sin (3\alpha/2) \overset{(6)}{=} (z^6 - 1) / 2iz^3,\quad \sin \alpha \overset{(6)}{=} (z^4 - 1) / 2iz^2,$$

$$\sin 2\alpha \overset{(6)}{=} (z^8 - 1) / 2iz^4,\quad \sin 3\alpha \overset{(6)}{=} (z^{12} - 1) / 2iz^6,$$

pa je:

$$\begin{aligned} \sin \alpha + \sin 2\alpha + \sin 3\alpha &= ((z^4 - 1) / 2iz^2) + ((z^8 - 1) / 2iz^4) + ((z^{12} - 1) / 2iz^6) \\ &= (z^{12} + z^{10} + z^8 - z^4 - z^2 - 1) / 2iz^6 \\ &= ((z^8 - 1) (z^4 + z^2 + 1) / 2iz^6)\cdot((z^2 - 1) / (z^2 - 1)) \\ &= (z^8 - 1) (z^6 - 1) / 2iz^6(z^2 - 1) \\ &= (((z^8 - 1) / 2iz^4)\cdot((z^6 - 1) / 2iz^3)) \;/\; ((z^2 - 1) / 2iz) \\ &= \mathbf{(\sin 2\alpha\cdot\sin (3\alpha/2)) / \sin (\alpha/2)} . \end{aligned}$$

Primjer 4. Riješiti trigonometrijsku jednadžbu

$$8 \cos^3 x - 6 \cos x + 2 = 0. \qquad (*)$$

Rješenje:

Neka je $\quad z = \cos x + i \sin x.$

Tada iz (*), primjenom (1), dobivamo:

$$\begin{aligned} 8\,((z^2 + 1) / 2z)^3 - 6\,((z^2 + 1) / 2z) + 2 &= 0 \\ z^6 + 2 z^3 + 1 &= 0 \\ (z^3 + 1)^2 &= 0 \\ z^3 + 1 &= 0 \\ (z + 1) (z^2 - z + 1) &= 0 \end{aligned}$$

$$z_1 = - 1 \quad (**) \qquad \vee \qquad z_{2,3} = (1/2) \pm (i\sqrt{3}/2) . \quad (***)$$

Iz (**) slijedi:

$$\cos x + i \sin x = - 1 \Rightarrow (\cos x = - 1 \;\cdot\; \sin x = 0) \Rightarrow$$

$$\Rightarrow \mathbf{x_1 = (2k + 1)\, \pi,\ k \in \mathbb{Z}.}$$